Temporal dynamics of incoherent waves in noninstantaneous response nonlinear Kerr media

B. Kibler,[1,*] C. Michel,[2] J. Garnier,[3] and A. Picozzi[1]

[1]*Laboratoire Interdisciplinaire Carnot de Bourgogne (ICB), CNRS UMR 6303—Université de Bourgogne, Dijon, France*

[2]*Laboratoire de Physique de la Matiére Condensée, CNRS UMR 7336—Université de Nice Sophia-Antipolis, Nice, France*

[3]*Ecole Normale Supérieure, Applied Mathematics Department, CNRS, 45 rue d'Ulm, Paris, France*

**Corresponding author: bertrand.kibler@u-bourgogne.fr*

Received March 21, 2012; revised May 3, 2012; accepted May 6, 2012;
posted May 7, 2012 (Doc. ID 164775); published June 19, 2012

We consider the temporal evolution of an incoherent optical wave that propagates in a noninstantaneous response nonlinear medium, such as single mode optical fibers. In contrast with the expected Raman-like spectral redshift due to a delayed nonlinear response, we show that a highly noninstantaneous response leads to a genuine modulational instability of the incoherent optical wave. We derive a Vlasov-like kinetic equation that provides a detailed description of this process of incoherent modulational instability in the temporal domain.

OCIS codes: 190.4370, 190.5650, 030.1640.

Nonlinear optics with incoherent waves is attracting a renewed interest since the first experimental observation of incoherent spatial solitons and incoherent modulational instability (MI) in photorefractive crystals [1–3]. Statistical nonlinear optics is now studied in various different circumstances, including wave propagation in homogeneous [4–8] or periodic media [9] or cavity systems [10–13]. In particular, the wave turbulence (WT) theory [14,15] proved efficient in describing important phenomena, such as the irreversible process of optical wave thermalization toward the thermodynamic equilibrium state [16]. As a remarkable consequence, a *classical* optical wave can exhibit a process of condensation [17,18], whose thermodynamic properties are analogous to those of quantum Bose–Einstein condensation, a feature which generated much interest [8,10,11,19]. The WT theory also provides a detailed description of the self-organization of incoherent optical waves into *nonequilibrium* stable structures, such as spectral incoherent solitons in noninstantaneous response nonlinear media [20,21], or incoherent solitons in highly nonlocal nonlinear media [22].

In this Letter we revisit the problem of the propagation of a temporally incoherent wave in a Kerr medium characterized by a noninstantaneous response time (e.g., optical fiber). Because of the causality property inherent to the nonlinear response function, the dynamics of the incoherent wave is known to be ruled by an antisymmetric gain curve—a well-known property in the example of the Raman effect. Then the low-frequency components of the incoherent field are amplified to the detriment of its high-frequency components, thus leading to a global spectral redshift of the incoherent wave.

Our aim in this Letter is to show that this general physical picture changes in a substantial way in the presence of a *highly* noninstantaneous nonlinear response of the material. In contrast with the expected spectral redshift, we show here that a highly noninstantaneous response leads to a genuine process of incoherent MI of the wave, which is characterized by the growth of two symmetric MI bands within the spectrum of the incoherent wave. Using the WT theory, we derive a Vlasov-like equation, whose self-consistent potential is constrained by the causality condition of the response function. The numerical simulations of the Vlasov equation are found in agreement with those of the nonlinear Schrödinger equation (NLSE). We can expect the experimental observation of this temporal incoherent MI thanks to the recent progress made on the fabrication of photonic crystal fibers (PCFs) filled with molecular liquids displaying highly noninstantaneous Kerr responses [23].

NLSE simulations. We consider the standard NLSE accounting for a noninstantaneous Kerr effect:

$$i\partial_z A = \beta\partial_{tt}A + \gamma A\int_{-\infty}^{+\infty} R(t-t')|A(z,t')|^2 \mathrm{d}t', \qquad (1)$$

where the response function $R(t)$ is constrained by the causality condition, $R(t) = 0$ for $t < 0$. The typical width of $R(t)$, say τ_R, denotes the response time of the nonlinearity. $\beta = -\beta_2/2$ refers to the second-order dispersion coefficient in the anomalous dispersion regime ($\beta > 0$), and γ is the nonlinear Kerr coefficient.

In the following we consider the regime in which nonlinear effects are of the same order as linear dispersive effects, $L_d \sim L_{nl}$, where $L_d = t_c^2/\beta$ and $L_{nl} = 1/(\gamma P)$ are the dispersive and nonlinear lengths, respectively, while t_c is the coherence time of the incoherent wave of average power P. The dynamics is ruled by the comparison of τ_R with the "healing time," $\tau_0 = (\beta L_{nl})^{1/2}$, which denotes the typical temporal MI period in the limit of an instantaneous nonlinear response ($\tau_R \to 0$).

In the regime $\tau_R \sim \tau_0$, one recovers the well-known process of spectral redshift of the incoherent wave: The antisymmetric spectral gain curve $g(\omega) = \Im[\tilde{R}(\omega)]$ [$\tilde{R}(\omega)$ being the Fourier transform of $R(t)$] leads to a transfer of energy from the high-frequency components toward the low-frequency components of the wave. It is in this regime that spectral incoherent solitons have been spontaneously generated by exploiting the Raman effect in

optical fiber systems [20,21]. By using the WT theory, these incoherent structures can be described in detail by a kinetic equation analogous to the weak Langmuir turbulence equation in plasma physics [20,21].

The system enters into a new regime when $\tau_R \gg \tau_0$, as illustrated in the numerical simulations of the NLSE [Eq. (1)] reported in Fig. 1. The initial condition is a partially coherent wave with a Lorentzian spectrum and random spectral phases, i.e., $A(z=0,t)$ is of zero mean and obeys a *stationary* Gaussian statistics. Note, however, that the statistics does not need to be Gaussian initially. Thanks to the highly noninstantaneous response, the field relaxes to a state of Gaussian statistics during the propagation [7]. We considered in this example an exponentiallike response function, $R(t) = H(t)t^2 \exp(-t/\tau_R)/(2\tau_R^3)$, where the prefactor t^2 avoids discontinuities in the derivative of $R(t)$ at $t=0$. However, we underline that the particular choice of the response function $R(t)$ and of the initial spectrum do not affect the generality of our results. We observe in Fig. 1 that, in marked contrast with the expected spectral redshift, the incoherent wave exhibits a genuine process of incoherent MI, as illustrated by the snake behavior of the spectrogram [see Figs. 1(a) and 1(b), while 1(c) will be discussed later]. Note that we considered here a high value of $\tau_R/\tau_0 = 100$ so as to guarantee a good agreement with the Vlasov theory; however, similar results are obtained for smaller response times (typically $\tau_R/\tau_0 > 10$).

We stress the *fundamental different nature of this incoherent MI with respect to the incoherent MI considered in instantaneous response Kerr media* [4]. Indeed, in the limit $\tau_R \to 0$, incoherent MI can only take place if the spectral width of the incoherent wave is smaller than the MI frequency, $\Delta\omega \ll \omega_{\mathrm{MI}}$ [4]. This means that temporal modulations associated to MI are more rapid than the time correlation, $t_c \gg \tau_0$, i.e., MI modulations take place within each individual fluctuation of the incoherent wave. This is in contrast with Fig. 1, where incoherent MI manifests itself by a slow modulation of the whole random wave profile, i.e., the modulation frequency is smaller than the spectral bandwidth, $\omega_{\mathrm{MI}} \ll \Delta\omega$.

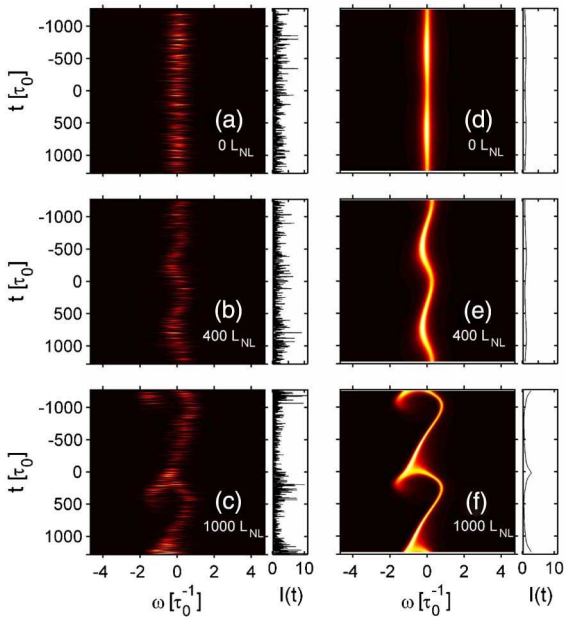

Fig. 1. (Color online) Numerical simulation of the NLSE [Eq. (1)] showing the evolution of the spectrograms of the wave at different propagation lengths (left column), and corresponding simulation of the Vlasov equation [Eq. (2)] with the same parameters ($\tau_R = 100$ [τ_0], $\Delta\omega/2\pi = 0.05$ [τ_0^{-1}]), except that the incoherent MI is seeded (right column). We underline the agreement between the NLSE and Vlasov simulations.

Vlasov approach. To describe this novel process of incoherent MI, we derive a Vlasov-like kinetic equation that governs the evolution of the *averaged* local spectrum of the wave, $n_\omega(t,z) = \int B(t,\tau,z)\exp(-i\omega\tau)\mathrm{d}\tau$, where $B(t,\tau,z) = \langle A(t+\tau/2,z)A^*(t-\tau/2,z)\rangle$ is the correlation function and $\langle . \rangle$ denotes an average over the realizations. Because of the highly noninstantaneous response $\varepsilon = \tau_0/\tau_R \ll 1$, we perform a multiscale series expansion with $B(t,\tau,z) = B^{(0)}(\varepsilon t,\tau,\varepsilon z)$ and $R(t) = \varepsilon R^{(0)}(\varepsilon t)$ (details will be reported elsewhere), which gives

$$\partial_z n_\omega(t,z) = 2\beta\omega\partial_t n_\omega(t,z) + \partial_t V(t,z)\partial_\omega n_\omega(t,z), \qquad (2)$$

with the self-consistent potential $V(t,z) = \gamma\int_{-\infty}^{+\infty} R(t-t')N(t',z)\mathrm{d}t'$, where $N(t,z) = B(t,\tau=0,z) = (2\pi)^{-1}\int n_\omega(t,z)\mathrm{d}\omega$ denotes the intensity profile.

The Vlasov equation [Eq. (2)] is of a different form than the conventional Vlasov equations used to describe nonlinear incoherent waves (see [3,5,7,15,24]). Its structure is in fact analogous to the Vlasov equation recently derived to describe highly nonlocal spatial effects [22]. In particular, the Vlasov equation [Eq. (2)] provides an "exact" statistical description of the incoherent wave (see [22]). Note, however, that contrary to the spatial case [22], in the temporal domain the self-consistent potential $V(t,z)$ is constrained by the causality property of $R(t)$, a feature which breaks the Hamiltonian structure of the Vlasov equation.

The MI analysis is performed by following the usual procedure (see, e.g., [3,5,7,24]). We perturb the statistical stationary solution of the Vlasov equation, $n_\omega(t,z) = n_\omega^0 + \epsilon_\omega(t,z)$, with $\epsilon_\omega(t,z) \ll n_\omega^0$, and linearize it to get $\partial_z\epsilon_\omega(t,z) - 2\beta\omega\partial_t\epsilon_\omega(t,z) - \gamma\partial_\omega n_\omega^0\int \mathrm{d}t'\partial_t R(t-t')\int \mathrm{d}\omega\epsilon_\omega(t',z) = 0$. This equation is solved by a Fourier–Laplace transform, $\tilde{\epsilon}_\omega(\Omega,p) = \int_0^\infty \mathrm{d}z\int_{-\infty}^{+\infty}\mathrm{d}t\exp(-pz-i\Omega t)\epsilon_\omega(t,z)$, where $\Re(p)$ denotes the MI growth rate. The dispersion relation reads

$$-1 = 2\gamma\beta\Omega^2\tilde{R}(\Omega)\int_{-\infty}^{+\infty}\frac{n_\omega^0}{(2\beta\Omega\omega + ip)^2}\mathrm{d}\omega. \qquad (3)$$

Assuming that the initial spectrum is Lorentzian shaped, $n_\omega^0 = P\Delta\omega/[\pi(\omega^2+\Delta\omega^2)]$, Eq. (3) gives

$$p(\Omega) = |\Omega|\left(\sqrt{2\beta\gamma P\tilde{R}(\Omega)} - 2\beta\Delta\omega\right), \qquad (4)$$

where the incoherent MI gain reads $g_{\mathrm{MI}}(\Omega) = 2\Re[p(\Omega)]$. Typical MI gain profiles are reported in Fig. 2. We remark in Fig. 2(a) that, as for the conventional incoherent MI [2–5,24], the finite bandwidth of the initial spectrum ($\Delta\omega \neq 0$) yields a (Landau) damping effect [3], which introduces a threshold for the incoherent MI, and significantly reduces the MI gain (g_{MI}) and

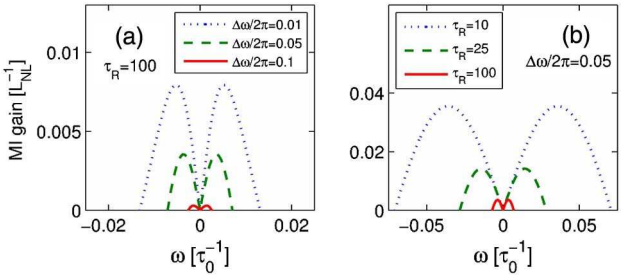

Fig. 2. (Color online) Plots of MI gain profiles [from Eq. (4)] for (a) different $\Delta\omega$ (in units of τ_0^{-1}) and (b) different τ_R (in units of τ_0).

the MI frequency (Ω_{MI}). Also note that $\Omega_{\mathrm{MI}} < \Delta\omega$, which confirms the fundamental difference with incoherent MI in instantaneous response nonlinear media [4]. We remark in Fig. 2(b) that g_{MI} and Ω_{MI} decrease as the response time τ_R increases, a feature that simply results from the scaling property of the Fourier transform $\tilde{R}(\Omega)$ of the response function. We recall that the validity of the Vlasov equation [Eq. (2)] is constrained by the multiscale expansion, $\varepsilon = \tau_0/\tau_R \ll 1$. As discussed above, in the regime $\tau_R \sim \tau_0$, the Vlasov equation breaks down and the appropriate kinetic description is provided by the weak Langmuir turbulence equation [20,21]. This equation does not describe any MI process, so that the incoherent MI discussed here is fundamentally related to the highly noninstantaneous regime, $\tau_0/\tau_R \ll 1$. We remark in this respect that, in the coherent case, MI takes place in both the weakly or highly noninstantaneous regimes, although the MI gain bands can exhibit some asymmetry [25]. Conversely, recalling that the functions $\Re[\tilde{R}(\Omega)]$ and $\Im[\tilde{R}(\Omega)]$ are, respectively, even and odd, the MI gain in Eq. (4) is always even, i.e., incoherent MI is characterized by the growth of two symmetric sidebands.

We report in Fig. 1 (right column) the numerical simulation of the Vlasov equation [Eq. (2)]. The initial condition is a stationary spectrum, n_ω^0, periodically perturbed to seed the incoherent MI with the maximum growth rate. We stress the good agreement with the unseeded MI simulation of the NLSE.

We can expect the observation of this incoherent MI considering a PCF filled with, e.g., carbon disulfide, where τ_R is of a few picoseconds [23]. In this case, the simulation in Fig. 1 would correspond to a power of few watts and a correlation time of the initial wave slightly below 1 ps.

In the spatial domain, incoherent MI was shown to eventually lead to the formation of stable incoherent soliton states [22]. Here, because of the causality of $R(t)$, the spectrum exhibits a redshift in the *nonlinear regime* of propagation [$z > 400L_{nl}$, Figs. 1(e) and 1(f)], i.e., beyond the linear regime described by the MI. This seems to prevent the formation of stable incoherent soliton structures. The long-term evolution of the novel process of incoherent MI reported here will be the subject of future investigations.

References

1. M. Mitchell, Z. Chen, M. Shih, and M. Segev, Phys. Rev. Lett. **77**, 490 (1996).
2. See, e.g., M. Soljacic, M. Segev, T. Coskun, D. N. Christodoulides, and A. Vishwanath, Phys. Rev. Lett. **84**, 467 (2000).
3. B. Hall, M. Lisak, D. Anderson, R. Fedele, and V. E. Semenov, Phys. Rev. E **65**, 035602 (2002).
4. A. Sauter, S. Pitois, G. Millot, and A. Picozzi, Opt. Lett. **30**, 2143 (2005).
5. D. Dylov and J. Fleischer, in *Localized States in Physics: Solitons and Patterns* (Springer, 2011), Part 1, pp. 17–34.
6. K. Hammani, B. Kibler, C. Finot, and A. Picozzi, Phys. Lett. A **374**, 3585 (2010).
7. J. Garnier, J.-P. Ayanides, and O. Morice, J. Opt. Soc. Am. B **20**, 1409 (2003).
8. U. Bortolozzo, J. Laurie, S. Nazarenko, and S. Residori, J. Opt. Soc. Am. B **26**, 2280 (2009).
9. Y. Silberberg, Y. Lahini, E. Small, and R. Morandotti, Phys. Rev. Lett. **102**, 233904 (2009).
10. C. Conti, M. Leonetti, A. Fratalocchi, L. Angelani, and G. Ruocco, Phys. Rev. Lett. **101**, 143901 (2008).
11. R. Weill, B. Fischer, and O. Gat, Phys. Rev. Lett. **104**, 173901 (2010).
12. S. Babin, D. Churkin, A. Ismagulov, S. Kablukov, and E. Podivilov, J. Opt. Soc. Am. B **24**, 1729 (2007).
13. E. G. Turitsyna, G. Falkovich, V. K. Mezentsev, and S. K. Turitsyn, Phys. Rev. A **80**, 031804 (2009).
14. V. E. Zakharov, V. S. L'vov, and G. Falkovich, *Kolmogorov Spectra of Turbulence I* (Springer, 1992).
15. For a simple introduction in optics, see, e.g., A. Picozzi, Opt. Express **15**, 9063 (2007).
16. P. Suret, S. Randoux, H. R. Jauslin, and A. Picozzi, Phys. Rev. Lett. **104**, 054101 (2010).
17. G. Düring, A. Picozzi, and S. Rica, Physica D **238**, 1524 (2009).
18. C. Sun, S. Jia, C. Barsi, S. Rica, A. Picozzi, and J. Fleischer, Nat. Phys., doi: 10.1038/NPHYS2278 (to be published).
19. A. Picozzi and M. Haelterman, Phys. Rev. Lett. **92**, 103901 (2004).
20. A. Picozzi, S. Pitois, and G. Millot, Phys. Rev. Lett. **101**, 093901 (2008).
21. B. Kibler, C. Michel, A. Kudlinski, B. Barviau, G. Millot, and A. Picozzi, Phys. Rev. E **84**, 066605 (2011).
22. A. Picozzi and J. Garnier, Phys. Rev. Lett. **107**, 233901 (2011).
23. C. Conti, M. A. Schmidt, P. St. J. Russell, and F. Biancalana, Phys. Rev. Lett. **105**, 263902 (2010).
24. A. Hasegawa, Phys. Fluids **18**, 77 (1975).
25. P. Béjot, B. Kibler, E. Hertz, B. Lavorel, and O. Faucher, Phys. Rev. A **83**, 013830 (2011).